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In–out propagator in de Sitter space from general boundary quantum field theory

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ABSTRACT

The general boundary formulation of quantum theory is applied to quantize a real massive scalar field in de Sitter space. The space–time region where the dynamics of the field takes place is bounded by one spacelike hypersurface of constant conformal de Sitter time. The computation of the amplitude in the presence of a linear interaction with a source field with compact support in the region considered provides the expression of the Feynman propagator which coincides with the so-called in–out propagator.

1. Introduction

The dynamics of a scalar field in de Sitter space is among the most studied quantum field theory in a gravitational background [1]. The main reason is due to the special role the de Sitter universe plays in modern cosmology. Measurements of the cosmic microwave background anisotropies [2] are consistent with an early inflationary stage of the universe that can be approximately described by the de Sitter geometry. Moreover, the current accelerated phase of expansion suggests de Sitter geometry as the one approached asymptotically by our universe.

From a quantum field theoretical point of view, the interest is devoted to the computation and analysis of the properties of n-point functions of the field. However, de Sitter space lacks a Killing vector which is globally timelike. Hence no prescription of a preferred vacuum state for the quantum field exists, and different vacua have been proposed, among them the Bunch–Davies vacuum [3] and a series of vacua called α vacua [4]. These states define different Green functions whose composition properties, as suggested by Polyakov in [5], are relevant for the stability of de Sitter space.

We present in this note the quantization of a real massive scalar field in de Sitter space within the general boundary formulation (GBF) of quantum theory, extending previous results [16,17]. The GBF is a new axiomatic formulation based on the mathematical framework of topological quantum field theory [6,7]. The set of axioms is meant to associate algebraic structures with geometrical ones and to guarantee the consistency of these structures. The main novelty of the GBF is the possibility of describing the quantum dynamics of fields in spacetime regions bounded by arbitrary hypersurfaces, therefore not restricted to the usual Cauchy ones. The acceptability of the GBF as a viable formulation of quantum field theory both in flat and curved spaces depends on (at least) two conditions: On the one hand the ability of the GBF to reproduce known results and to offer a new perspective on them, and on the other hand to propose solutions to open problems present in other formulations of quantum theory. With respect to this second point, it is well known that a major obstruction for the implementation of the standard S-matrix technique in Anti-de Sitter space (AdS) is the lack of temporal asymptotic free states. A proposal [8] in terms on timelike boundary sates has been advanced in the context of AdS/CFT correspondence [9]. Remarkably, the GBF offers a first principle approach to consistently define states on timelike hypersurface, [10]. In particular in Minkowski spacetime, a so-called hypercylinder region, namely a three-ball in space extended over all of time, was considered in [11,12] and the asymptotic amplitude of states defined on the boundary of the hypercylinder was shown to correspond to the standard S-matrix in Minkowski. An analogous hypercylinder region was the basis for proposing an S-matrix for spatially asymptotic states in AdS, [13,14].

This letter is concerned with the first point mentioned above. In particular we show how to obtain, within the GBF, the expression of the in–out Feynman propagator for a scalar field in de Sitter space. A derivation of this propagator based on standard techniques is presented in details in [15], to which we will refer for comparing our results. In Section 2 we give a minimal presentation of the GBF introducing the algebraic structures relevant in the

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succeeding sections. Section 3.1 recalls from [17] the quantization of a real massive scalar field in a space–time region bounded by two spacelike hypersurfaces labeled by constant values of the conformal de Sitter time coordinate. In Section 3.2 we present our main result: The derivation of the in–out Feynman propagator obtained by choosing for the quantization region the one bounded by one spacelike hypersurface. Conclusions are offered in Section 4.

2. GBF in Lorentzian spacetime

Two quantization prescriptions have been so far developed for the GBF: The Schrödinger–Feynman one, where path integral quantization is combined with the Schrödinger representation for quantum states, and the holomorphic quantization [18] inspired by geometric quantization techniques. The quantizations were shown to be equivalent [19]. We adopt in the following the Schrödinger–Feynman quantization scheme.

Consider a real scalar field ϕ with mass m described by the action

$$S_M[\phi] = -\frac{1}{2}\int_M d^4x\sqrt{-g}\left(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + m^2\phi^2\right) \tag{1}$$

in a spacetime region M of a 4-dimensional Lorentzian manifold $(\mathcal{M}, g)$. The GBF axioms associate a space of states $\mathcal{H}_{\partial M}$ with the boundary ∂M of the region, and an amplitude map $\rho_M : \mathcal{H} \to \mathbb{C}$ to this region, defined for a state $\psi \in \mathcal{H}$ as

$$\rho_M(\psi) = \int \mathcal{D}\varphi\, \psi(\varphi) Z_M(\varphi), \tag{2}$$

where φ is a boundary field configuration on ∂M and Z_M denotes the field propagator

$$Z_M(\varphi) = \int \mathcal{D}\phi\, e^{\mathrm{i}S_M[\phi]}, \tag{3}$$

where the integral is extended over all the configurations ϕ that reduce to φ on the boundary ∂M. The field propagator can be expressed in terms of the action of classical solution ϕ_{cl} matching the boundary field configuration on ∂M, i.e. $\phi_{cl}|_{\partial M} = \varphi$, as

$$Z_M(\varphi) = N_M e^{\mathrm{i}S_M[\phi_{cl}]}, \tag{4}$$

where N_M is a normalization factor fixed by requiring that the amplitude of the vacuum state $\psi_0 \in \mathcal{H}_{\partial M}$ equals one. In the presence of a linear interaction with a source field μ with compact support in the region M, in the form $\int d^4x\,\mu(x)\phi(x)$, the amplitude (2) acquires the following structure

$$\rho_{M,\mu}(\psi_\eta) = \rho_M(\psi_\eta)\exp\left(\mathrm{i}\int dx\,\mu(x)\hat{\eta}(x)\right) \times \exp\left(\frac{i}{2}\int dx\,dx'\,\mu(x)G_F(x,x')\mu(x')\right), \tag{5}$$

where ψ_η denotes a coherent state, determined by the complex function η as

$$\psi_\eta(\varphi) = K_\eta \exp\left(\int d^3x\,\eta(\underline{x})\varphi(\underline{x})\right)\psi_0(\varphi), \tag{6}$$

K_η being a normalization factor, ρ_M denotes the amplitude on the coherent state ψ_η in the case of the free theory, i.e. $\rho_M(\psi_\eta) = \rho_{M,\mu=0}(\psi_\eta)$, $\hat{\eta}$ is a complex solution of the equations of motion that depends on the function η, and G_F is the Feynman propagator of the field in the region considered. In the case where the boundary of the region M is given by the disjoint union of two Cauchy surfaces, at an initial time t_{in} and a final time t_{out}, the state on the boundary factorizes as the tensor product $\psi = \psi_{in} \otimes \psi_{out} \in \mathcal{H}_{in} \otimes \mathcal{H}^*_{out}$, with $\mathcal{H}_{in}$ and $\mathcal{H}^*_{out}$ being the state spaces associated with the Cauchy surface at t_{in} and t_{out} respectively (the star denoting the dual space). When the initial and final times are sent to the past and future infinities, it has been shown that the amplitude (5) reproduces the standard S-matrix (for an interaction given by the linear term $\int d^4x\,\mu(x)\phi(x)$)[1] in Minkowski [11,12], Euclidean [21], de Sitter [16,17] and Rindler space [22]. Finally, it is worth mentioning that a consistent probability interpretation, generalizing the standard Born rule, has been developed for the amplitude (2), see [20].

3. GBF in de Sitter space

We work with the coordinate system $(t, \underline{x})$ such that $t \in (0, \infty)$, $\underline{x} \in \mathbb{R}^3$ which covers only the half of de Sitter space (t is the conformal time),

$$ds^2 = \frac{R^2}{t^2}\left(dt^2 - d\underline{x}^2\right), \tag{7}$$

where R denotes the inverse of the Hubble constant. We are interested in two spacetime regions. The first one is a finite time-interval region bounded by the disjoint union of two hypersurfaces of constant conformal time t, namely $\Sigma_1 = \{(t, \underline{x}) : t = t_1\}$ and $\Sigma_2 = \{(t, \underline{x}) : t = t_2\}$, with $t_1 < t_2$. We denote this spacetime region, $M = [t_1, t_2] \times \mathbb{R}^3$, by the subscript $[t_1, t_2]$. The second region is bounded by one hypersurface of constant time, say Σ_2. This region can be seen as the limit for $t_1 \to 0$ of region $M_{[t_1,t_2]}$ and will be denoted by M_{t_2}.

3.1. Feynman propagator in region $M_{[t_1,t_2]}$

In [16,17] the GBF was constructed for a real massive scalar field in region $M_{[t_1,t_2]}$ both in the case of the free theory and the general interacting one. The classical solution of the Klein–Gordon equation reducing to the boundary field configurations φ_i at t_i $(i = 1, 2)$, can be expressed in the form

$$\phi(t, \underline{x}) = \left(\frac{\delta_k(t, t_2)}{\delta_k(t_1, t_2)}\varphi_1\right)(\underline{x}) + \left(\frac{\delta_k(t_1, t)}{\delta_k(t_1, t_2)}\varphi_2\right)(\underline{x}), \tag{8}$$

where the quotients have to be understood as operators acting on a Fourier expansion of the boundary configurations φ_1 and φ_2,

$$\varphi_i(\underline{x}) = \int \frac{d^3k}{(2\pi)^3} e^{\mathrm{i}\underline{k}\cdot\underline{x}}\varphi(\underline{k}), \quad i = 1, 2 \tag{9}$$

and the operator δ_k is defined as

$$\delta_k(z, \hat{z}) := (z\hat{z})^{3/2}\left[J_\nu(kz)\,Y_\nu(k\hat{z}) - Y_\nu(kz)\,J_\nu(k\hat{z})\right], \tag{10}$$

with $k = |\underline{k}|$ and J_ν and Y_ν denoting the Bessel functions of the first and second kind respectively, and $\nu = \sqrt{\frac{9}{4} - (mR)^2}$.

The boundary Hilbert space is given by the tensor product $\mathcal{H}_1 \otimes \mathcal{H}^*_2$ where $\mathcal{H}_i$ is the Hilbert space associated the boundary Σ_i $(i = 1, 2)$. In $\mathcal{H}_i$ the vacuum state corresponding to the Bunch–Davies vacuum takes the form

$$\psi_{0,i}(\varphi_i) = C_i \exp\left(\frac{\mathrm{i}}{2}\int d^3x\,\varphi_i(\underline{x})\,(A_i\varphi_i)(\underline{x})\right) \tag{11}$$

[1] The case of a general interacting theory can be implemented perturbatively using functional derivative techniques as presented in [11,12].

where C_i is the normalization factor and A_i the operator

$$A_i = \frac{R^2}{t_i^2}\left[k\frac{\left(H_\nu^{(1)}(kt_i)\right)'}{H_\nu^{(1)}(kt_i)} + \frac{3}{2t_i}\right], \tag{12}$$

$H_\nu^{(1)}$ being the Hankel function and the prime denoting derivative w.r.t. the argument. The action evaluated on the classical solution (8) and the vacuum state (11) determine the field propagator $Z_{[t_1,t_2]}(\varphi_1,\varphi_2)$.

The asymptotic limit of the amplitude (5), namely the amplitude computed in the presence of the interacting term $\int d^4x\,\mu(x)\phi(x)$ where the source field μ has compact support in the interior of $M_{[t_1,t_2]}$, was interpreted as the S-matrix for the scalar field. The Feynman propagator appearing in (5) coincides with the in–in propagator of [15] (see eq. (184)),

$$G_F(x,x') = \mathrm{i}\frac{(tt')^{3/2}}{8\pi R^2}\int_0^\infty dk\,k\frac{\sin(k|\underline{x}-\underline{x}'|)}{|\underline{x}-\underline{x}'|} \times H_\nu^{(1)}(kt_<)H_\nu^{(2)}(kt_>), \tag{13}$$

where $t_> = \max(t,t')$ and $t_< = \min(t,t')$.

3.2. Feynman propagator in region M_{t_2}

In the region M_{t_2}, solutions of the Klein–Gordon equation are obtainable by taking the limit $t_1 \to 0$ of expression (8) and imposing that the field vanishes at $t_1 = 0$, namely the boundary condition $\varphi_1 = 0$. The result is expressed in terms of the Bessel function of the first kind only, because of the singular behavior of the Bessel function of the second kind for small arguments [23], and results to be

$$\phi(t,\underline{x}) = \left(\frac{t^{3/2}J_\nu(kt)}{t_2^{3/2}J_\nu(kt_2)}\varphi_2\right)(\underline{x}). \tag{14}$$

The action for this classical solution in terms of the boundary field configuration φ_2 reads as

$$S_{M_{t_2}}(\phi) = \frac{1}{2}\int d^3\underline{x}\,\varphi_2(\underline{x})\,(W\varphi_2)(\underline{x}), \tag{15}$$

where the operator W is given by

$$W = \frac{R^2}{t_2^2}\left(\frac{3}{2t_2} + k\frac{J_\nu'(kt_2)}{J_\nu(kt_2)}\right). \tag{16}$$

In the boundary Hilbert space $\mathcal{H}_2$ the vacuum state is given by (11) with $i=2$. The resulting field propagator $Z_{t_2}(\varphi_2)$ satisfies the composition rule

$$Z_{t_3}(\varphi_3) = \int \mathcal{D}\varphi_2\, Z_{[t_2,t_3]}(\varphi_2,\varphi_3)Z_{t_2}(\varphi_2). \tag{17}$$

The amplitude for a coherent state $\psi_{\eta,t_2} \in \mathcal{H}_2$, in the interaction picture as defined in [17], results to be

$$\rho_{M_{t_2}}(\psi_{\eta,t_2}) = \exp\left(\frac{\pi}{8R^2}\int d^3x\left[|\eta(\underline{x})|^2 - \eta^2(\underline{x})\right]\right), \tag{18}$$

which is independent of the conformal time t_2. By introducing in the action an interaction of the form $\int d^4x\,\mu(x)\phi(x)$, where the field μ has compact support in the interior of the region M_{t_2}, the amplitude for the coherent state ψ_{η,t_2} takes the form (5) with

$$\hat{\eta}(t,\underline{x}) = \frac{\pi}{2R^2}\left(J_\nu(kt)\eta\right)(\underline{x}), \tag{19}$$

and the Feynman propagator $G_F(x,x')$ given by

$$G_F(x,x') = \mathrm{i}\frac{(tt')^{3/2}}{4\pi R^2}\int_0^\infty dk\,k\frac{\sin(k|\underline{x}-\underline{x}'|)}{|\underline{x}-\underline{x}'|} \times J_\nu^{(1)}(kt_<)H_\nu^{(2)}(kt_>), \tag{20}$$

which coincides with the expression of the in–out propagator derived in [15] (see eq. (182)). The amplitude for this interacting theory turns out to be independent of the conformal time t_2 and in the asymptotic limit $t_2 \to \infty$ it can be interpreted as the S-matrix for the asymptotic coherent state $\psi_{\eta,\infty}$.

4. Conclusions

The present result extends previous ones obtained in [16,17] for a scalar field in de Sitter space, and has a clear relevance for the GBF. The derivation of the in–out propagator in de Sitter can be interpreted as a test for the ability of GBF to reproduce known results in quantum field theory in curved space. Moreover, the GBF offers a new perspective on this derivation: The in–in and in–out propagators have been obtained following three steps: Fixing the space–time regions of interest, solving the equation of motion in these regions and finally applying the Schrödinger–Feynman quantization prescription of the GBF to compute amplitudes for the free theory and the interacting one. In particular no limiting procedure is needed: The in–in propagator is obtained in the finite time-interval region bounded by two equal-time hypersurfaces and the in–out propagator in the region by only one equal-time hypersurface. No asymptotic limit that sends the boundary hypersurfaces to the past and future boundaries of de Sitter space has been implemented, in contrast to the treatment presented by Fukuma et al. in [15].

It will be interesting to investigate, for a quantum field in a curved space, the implementability within the GBF of the in–in formalism originally due to Schwinger [24] and further developed in [25]. The result presented here can be viewed as a first step in the construction of the general theory.

In [16,17] the quantum theory in the finite time-interval region was related to the so-called hypercylinder region enclosed by one connected timelike hypersurface defined a constant value of the radial coordinate. In particular it has been shown that the Hilbert spaces at the asymptotic boundary of the two regions are isomorphic. The question whether the quantum theory in the region M_{t_2} corresponds to the one in a different region (possibly a certain type of hypercylinder region) is open.

We shall elaborate on these two points elsewhere.

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